Math Notation

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1 Introduction

We provide a brief summary of the mathematical notation used in the course. We assume that you have seen the notation or are at least familiar with the concepts from other courses.

2 Sets

A set is a collection of objects. One interesting set is the *null set* or the *empty set*, the set that has no elements in it. We will refer to this set (denote this set) by the symbol $\emptyset$. We will use the notation

$$A = \{x : \ldots\}$$

to denote the set named A which consists of all the sets x that satisfy some property indicated by the "$\ldots$".

If A and B are sets, then $A \cup B$, the *union* of A and B, is the set consisting of all the elements from either A or B or both. The set $A \cap B$, the *intersection* of A and B is the collection of elements in both A and B symultaneously. If $e \in A$ (e is an element of the set A), then $\{e\}$ is the set consisting of the single element e and $A \setminus \{e\}$ is the set consisting of all the elements of A except the element e.

If A and B are sets, then if every element of B is also in A, we say that B is a *subset* of A. We write this as $A \subseteq A$. If C has elements not in A then we can write $C \not\subseteq A$.

If A, B and C are sets, the set $A \times B$ is the set of all ordered pairs of elements (a, b) where $a \in A$ and $b \in B$. Using our set notation described above we have:

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$

We can, of course, extend this notation to for example to $A \times B \times C$ to ordered triples (a, b, c) and beyond. $A \times B$ is sometimes called the *crossproduct* of A and B.

If A is a finite set then the number of elements in A is sometimes called the *cardinality* of A. We abbreviate this by the notation $|A|$.

If A is a set then the *powerset* of A is the collection (set) of all subsets of the set A, including the empty set and the set A itself. We denote this either by the functional notation $\mathcal{P}(A)$ or the exponential notation 2^A. The reason for this notation is that if A is a finite set, then we have the result

$$|2^A| = 2^{|A|}$$

which is an interesting notational convenience.

Another interesting result of our notation is that if A and B are finite sets then

$$|A \times B| = |A| \times |B|$$

where the second "$\times$" is normal integer multiplication.

3 Relations and Functions

If A and B are sets then a *relation* R between A and B is a subset of $A \times B$, $R \subseteq A \times B$. The first set, A in our example, is called the *domain* of the relation. The other set, B in our example, is called the *range* of the relation.

If each element of A appears exactly once in the relation R, such a relation is called a function. We might describe this as:

$$R : A \to B$$

For each element of A we can associate a single element of B via the releation R. If $(a, b) \in$ function R, then we can write $R(a) = b$.

If R is a relation, $R \subset A \times B$, we can define the inverse relation R^{-1} by

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

Note that if A is the domain of R and B is its range, then B is the domain of R^{-1} and A is its range.

If both R and R^{-1} are functions then for each element of A, R associates a unique element of B and for each element of B, R^{-1} associates a unique element of A. Another way of seeing this is that each element of the domain is the first element of exactly one pair in the relation and each element of the range is the second element in exactly one pair in the relation. If R and R^{-1} are both functions then R is said to be $1 \leftrightarrow 1$, read *1-to-1*.

3.1 Example

The relation R^{-1} always exists, but it may not always be a function, even if R is. The following very simple example illustrates this.

Consider the relation (described in pairs)

$$R = \{(1, b), (2, a), (3, a)(4, c)\}$$

The domain of R is $\{1, 2, 3, 4\}$ and the range of R is $\{a, b, c\}$. Note that R is a function, because for each element in the domain, there is exacly one pair that contains that element of the domain. Note that $R^{-1} = \{(b, 1), (a, 2), (a, 3), (c, 4)\}$ is not a function since there are two pairs that contain the first element a.

Since R^{-1} is not a function, R, in this case, is not $1 \leftrightarrow 1$ (which we could write as $1 \not\leftrightarrow 1$).

4 More on Sets

Suppose X and Y are sets and $a \in (X \cup Y)$. Let $R \subseteq X \times Y$. Then we can define $R \setminus a$ as

$$R \setminus \{a\} = R \cup ((X \setminus \{a\}) \times (Y \setminus \{a\}))$$

This is the collection of all the elements of R that do not have the element A as either a first or second element.

5 Sequences

5.1 Standard Approach

Suppose $A \subseteq \mathcal{X}$. We can define a sequence S as an ordered list of elements of A. We will denote this list by:

$$S = < a_1, a_2, a_3, \ldots, a_n >$$

This sequence has exactly n elements. The first element of the sequence is a_1, the second a_2, etc., with the last a_n. We will say that the sequence S is **based** on the elements of the set A, that is the elements of the sequence are taken from the elements of A. For reasons that will become apparent below, we may also call A the **range** of the sequence S.

We distinguish sets from sequences by using the delimeters $<$ and $>$ for sequences. The elements of of the sequence do not need to be distinct. For example, if $a,\ b \in A$, then the sequence

$$\overbrace{< a, a, a, \ldots, a >}^{\text{n times}}$$

of n a's is a perfectly fine sequence as is

$$< \overbrace{a, a, a, \ldots a}^{\text{n times}}, \overbrace{b, b, b, \ldots, b}^{\text{m times}} >$$

We will assume that our sequences are always indexed starting from 1, unless we state otherwise.

Most of our sequences will be finite, that is there will be a finite number of elements in the sequences.

5.2 An alternative approach

We can use an alternative approach to describe sequences. Suppose $\mathbb{Z}_n^+$ is the set of positive numbers starting at 1 and continuing up to and including n.[1]

Suppose we some set $A \subseteq \mathcal{X}$, then we can define the sequence S as a function

$$S:\ \mathbb{Z}_n^+ \rightarrow A$$

If we want to map to our other notation, the sequence would look like:

$$< S(1), S(2), S(3), \ldots, S(n) >$$

In this case we have $S(1) = a_1$, $S(2) = a_2$, etc.. Since S is a function, we could have the same $a \in A$ as the value of S for more than one number. Using this notation, it is easy to pick out specific elements of the sequence.

Of course, if we want to talk about infinite sequences, then our domain becomes $\mathbb{Z}^+$, the set of all positive integers.

[1] Normally $\mathbb{Z}_n$ is the set of numbers starting at 0. I have added the $+$ superscript above to indicate that we are not including 0.

5.3 Length of a Sequence

We will say that a sequence $S =< a_1, a_2, \ldots, a_n >$ is of length n. Alternatively if $S : \mathbb{Z}_n^+ \rightarrow A$ then this also represents a sequence of lenght n. In either case, we may write $len(S) = n$.

6 Relationships among sequences

6.1 Initial sequence

Suppose $S_1 =< a_1, a_2, a_3, \ldots, a_k >$ and $S_2 =< b_1, b_2, b_3, \ldots, b_m >$. We will say that S_1 is an **initial sequence** of S_2 if $k \leq m$ and $a_i = b_i$ for all $1 \leq i \leq k$. We might also call this an **initial subsequence**.

6.2 Subsequences

There are two notions of subsequence that we can define. Again, suppose we have $S_1 =< a_1, a_2, a_3, \ldots, a_k >$ and $S_2 =< b_1, b_2, b_3, \ldots, b_m >$. We will say that S_1 is a **proper subsequence** of S_2 if $k \leq m$ and there is some j, $1 \leq j \leq m$ such that $j + k - 1 \leq m$ and $a_1 = b_j$, $a_2 = b_{j+1}$, …, $a_k = b_{j+k-1}$.

What we are trying to say here is that S_1 appears exactly (with no additions and no gaps) inside S_2.

The other notion of subsequence is a bit more complicated to explain. Again, we start with S_1 and S_2 as defined above. In this case we want to be able to pick out the sequence S_1 from inside the elements of S_2. The elements of S_1 are in S_2 in the right order, but two elements that are adjacent in S_1 may not be adjacent in S_2. If this is the case, we will say that S_1 is just a **subsequence** of S_2.

To describe this more formally, we will say that S_1 is a **subsequence** of S_2 if there is an injection[2], $\mathcal{J}$,

$$\mathcal{J} : \mathbb{Z}_k^+ \rightarrow \mathbb{Z}_m^+$$

that maps the indicies of S_1 into the indicies of S_2 so that if $i < j$ then $\mathcal{J}(i) < \mathcal{J}(j)$ and $a_i = b_{\mathcal{J}(i)}$ for all $1 \leq i \leq k$. As a consequence of this definition, every element of the sequence S_1 appears in the sequence S_2 in the right order.

[2]Recall that an injection is one-to-one mapping that is not necessarily onto.

6.3 Sets of Sequences

We will find it useful to describe sets of sequences. If A is some base set then we can define

$$\mathcal{A} = \{< a_1, a_2, a_3, \ldots, a_k >: k \in \mathbb{Z}^+, a_i \in A, 1 \leq i \leq k\}$$

This is the set of all sequences using the elements from the base set A. We could limit this to some set of finite sequences as follows,

$$\mathcal{A}_n = \{< a_1, a_2, a_3, \ldots, a_k >: k \in \mathbb{Z}_n^+, a_i \in A, 1 \leq i \leq k\}$$

We may sometimes abbreviate/abuse the notation as follows:

$$\{< a_1, a_2, a_3, \ldots, a_k >: k \in \mathbb{Z}^+, a_i \in A, 1 \leq i \leq k\} = \{< a : a \in A >\} =< a : a \in A >$$

These are all the sequences of length $\leq n$ from the base set A.

If $\mathcal{A}$ and $\mathcal{A}_n$ are defined as above then it is easy to see that

$$\mathcal{A}_n \subset \mathcal{A}$$

and that if $k \leq n$ then every element (sequence) of $\mathcal{A}_k$ is an initial subsequence of some sequence in $\mathcal{A}_n$. In fact we could show that

$$\mathcal{A}_k \subset \mathcal{A}_n$$

6.4 Examples

Suppose our set $A = \{x, y, z, w\}$. Suppose we have

$$S =< x, y, z, w, x, x > .$$

The we might talk of $S(1) = x$, $S(2) = y$, $S(3) = z$, $\ldots$, $S(6) = x$.

If we define $S_1 =< x, y, z >$, then it should be clear that S_1 is an **initial subsequence**, a **proper subsequence** and a **subsequence** of S. The first two should be straighforward. For the **subsequence** property, we need to find the injection. In this case, it should be very easy:

$$\begin{array}{ccc} \mathbb{Z}_3^+ & \rightarrow & \mathbb{Z}_6^+ \\ 1 & \rightarrow & 1 \\ 2 & \rightarrow & 2 \\ 3 & \rightarrow & 3 \end{array}$$

If $S_2 =< y, w, x >$ then S_2 is a **subsequence** of S in two different ways. In the first way, the injection is:

$$\begin{array}{ccc} \mathbb{Z}_3^+ & \to & \mathbb{Z}_6^+ \\ 1 & \to & 2 \\ 2 & \to & 4 \\ 3 & \to & 5 \end{array}$$

Note here that the x in S_2 maps to the first x in S.

The other injection has a very similar form:

$$\begin{array}{ccc} \mathbb{Z}_3^+ & \to & \mathbb{Z}_6^+ \\ 1 & \to & 2 \\ 2 & \to & 4 \\ 3 & \to & 6 \end{array}$$

but in this case, the x in S_2 maps to the second x in S.

6.5 Exersizes

Suppose that $A = \{x, y, z, w\}$. How many elements (sequences) are there in $\mathcal{A}_3$? Be sure to justify your arguement.